

# A GEOMETRIC INTERPRETATION OF STANLEY'S MONOTONICITY THEOREM

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ABSTRACT. We present a new geometric proof of Stanley's monotonicity theorem for lattice polytopes, using an interpretation of  $\delta$ -polynomials of lattice polytopes in terms of orbifold Chow rings.

## 1. INTRODUCTION

Let  $P$  be a  $d$ -dimensional lattice polytope in a lattice  $N$  of rank  $n$ . That is,  $P$  is the convex hull of finitely many points in  $N \cong \mathbb{Z}^n$ . If  $m$  is a positive integer, then let  $f_P(m) := \#(mP \cap N)$  denote the number of lattice points in the  $m$ 'th dilate of  $P$ . A famous theorem of Ehrhart [6] asserts that  $f_P(m)$  is a polynomial in  $m$  of degree  $d$ , called the *Ehrhart polynomial* of  $P$ . The generating series of  $f_P(m)$  can be written in the form

$$\frac{\delta_P(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} f_P(m) t^m,$$

where  $\delta_P(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$  is a polynomial of degree at most  $d$  with integer coefficients, called the  $\delta$ -*polynomial* of  $P$ . Using techniques from commutative algebra, Stanley proved that the coefficients  $\delta_i$  are non-negative [12] and proved that  $\delta$ -polynomials of lattice polytopes satisfy the following monotonicity property [13, Theorem 3.3]. An alternative combinatorial proof of these results was given by Beck and Sottile in [2]. If  $f(t) = \sum_i f_i t^i$  and  $g(t) = \sum_i g_i t^i$  are polynomials with integer coefficients, then we write  $f(t) \leq g(t)$  if  $f_i \leq g_i$  for all  $i \geq 0$ .

**Theorem 1.1** (Stanley's Monotonicity Theorem). *If  $Q \subseteq P$  are lattice polytopes, then  $\delta_Q(t) \leq \delta_P(t)$ .*

We now present a new geometric proof of Stanley's theorem. We first recall the following geometric interpretation of  $\delta$ -polynomials of lattice polytopes. After replacing  $N$  with its intersection with the affine span of  $P$ , we may assume that  $N$  has rank  $d$ . Let  $\mathcal{T}$  be a regular, lattice triangulation of  $P$  and let  $\sigma$  denote the cone over  $P \times \{1\}$  in  $N_{\mathbb{R}} \times \mathbb{R}$ , where  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ . The triangulation  $\mathcal{T}$  induces a simplicial fan refinement  $\Delta$  of  $\sigma$ , with cones given by the cones over the faces of  $\mathcal{T}$ , and we may consider the  $(d+1)$ -dimensional, simplicial toric variety  $Y = Y(\Delta)$  associated to  $\Delta$ . The toric variety  $Y$  is *semi-projective* in the sense that it contains a torus-fixed point and is projective over its affinisation  $Y(\sigma)$  [8]. The cohomology ring  $H^*(X, \mathbb{Q})$  of a semi-projective, simplicial toric variety  $X$  was computed by Hausel and Sturmfels in [8], and it was observed by Jiang and Tseng [9,

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Lemma 2.7] that Hausel and Sturmfel's proof, along with the results in [7, Section 5.1], imply that  $H^*(X, \mathbb{Q})$  is isomorphic to the Chow ring  $A^*(X, \mathbb{Q})$ .

The orbifold Chow ring of a Deligne-Mumford stack was introduced by Abramovich, Graber and Vistoli [1] as the algebraic analogue of Chen and Ruan's orbifold cohomology ring [5]. Borisov, Chen and Smith introduced the notion of a toric stack in [4] and showed that any simplicial, semi-projective toric variety  $X$  has the canonical structure of a Deligne-Mumford stack. The orbifold Chow ring  $A_{\text{orb}}^*(X, \mathbb{Q})$  of  $X$  is a  $\mathbb{Q}$ -graded  $\mathbb{Q}$ -algebra and was computed by Jiang and Tseng in [9], generalising results in [4] (Remark 2.2). The following combinatorial observation follows from [14, Theorem 4.6] (c.f. [10, Corollary 1.2]).

$$(1) \quad \delta_P(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A_{\text{orb}}^i(Y, \mathbb{Q}) t^i.$$

If  $Q$  is a lattice polytope contained in  $P$ , then let  $N'$  denote the intersection of  $N$  with the affine span of  $Q$  and let  $\sigma'$  denote the cone over  $Q \times \{1\}$  in  $(N')_{\mathbb{R}} \times \mathbb{R}$ . One verifies that we may choose a regular, lattice triangulation  $\mathcal{T}$  of  $P$  which restricts to a regular, lattice triangulation of  $Q$ . In this case, the fan  $\Delta$  refining  $\sigma$  restricts to a fan  $\Sigma$  refining  $\sigma'$  and we may consider the semi-projective toric variety  $Y' = Y'(\Sigma)$ . The inclusion of  $N'$  in  $N$  induces a locally closed toric immersion  $Y' \hookrightarrow Y$  and a restriction map between the corresponding orbifold Chow rings. We will prove the following lemma in Section 2.

**Lemma 1.2.** *The morphism  $Y' \hookrightarrow Y$  induces a surjective graded ring homomorphism  $A_{\text{orb}}^*(Y, \mathbb{Q}) \rightarrow A_{\text{orb}}^*(Y', \mathbb{Q})$ .*

By (1),  $\delta_P(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A_{\text{orb}}^i(Y, \mathbb{Q}) t^i$  and  $\delta_Q(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A_{\text{orb}}^i(Y', \mathbb{Q}) t^i$ , and we conclude that  $\delta_Q(t) \leq \delta_P(t)$ , as desired.

*Remark 1.3.* If we regard the empty face as a face of the triangulation  $\mathcal{T}$  of dimension  $-1$ , then the  $h$ -vector of  $\mathcal{T}$  is defined by

$$h_{\mathcal{T}}(t) = \sum_F t^{\dim F + 1} (1 - t)^{d - \dim F},$$

where the sum ranges over all faces  $F$  in  $\mathcal{T}$ . It is a well known fact that  $0 \leq h_{\mathcal{T}}(t) \leq \delta_P(t)$  and  $h_{\mathcal{T}}(t) = \delta_P(t)$  if and only if  $\mathcal{T}$  is a unimodular triangulation [3, 11]. We have the following geometric interpretation of this result.

It follows from the definition of the orbifold Chow ring (see Section 2) that  $A^*(Y, \mathbb{Q})$  is a direct summand of  $A_{\text{orb}}^*(Y, \mathbb{Q})$  and  $A^*(Y, \mathbb{Q}) = A_{\text{orb}}^*(Y, \mathbb{Q})$  if and only if  $Y$  is smooth. The result now follows from the fact that  $h_{\mathcal{T}}(t) = \sum_{i \geq 0} \dim_{\mathbb{Q}} A^i(Y, \mathbb{Q}) t^i$  [8, Corollary 2.12] and the fact that  $Y$  is smooth if and only if  $\mathcal{T}$  is a unimodular triangulation.

All varieties and stacks will be over the complex numbers. In Section 2, we will review orbifold Chow rings and prove Lemma 1.2.

## 2. ORBIFOLD CHOW RINGS

The orbifold Chow ring  $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$  of a Deligne-Mumford stack  $\mathcal{X}$  was introduced by Abramovich, Graber and Vistoli as the degree 0 piece of the small quantum cohomology ring of  $\mathcal{X}$  [1]. We will review the structure of  $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$  as a  $\mathbb{Q}$ -graded vector space and refer the reader to [1] for the relevant details and the description of the ring structure of  $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$ . The *inertia stack*  $\mathcal{IX}$  of  $\mathcal{X}$  is a Deligne-Mumford stack whose objects consist of pairs  $(x, \alpha)$ , where  $x$  is an object of  $\mathcal{X}$  and  $\alpha$  is an automorphism of  $x$ . If  $\mathcal{X}_1, \dots, \mathcal{X}_r$  denote the connected components of  $\mathcal{IX}$ , then

$$A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q}) = \bigoplus_{j=1}^r A^*(|\mathcal{X}_j|, \mathbb{Q})[s_j],$$

where  $|\mathcal{X}_j|$  is the coarse moduli space of  $\mathcal{X}_j$ ,  $s_j \in \mathbb{Q}$  is the *age* of  $\mathcal{X}_j$  and  $[s_j]$  denotes a grading shift by  $s_j$ . If we identify  $\mathcal{X}$  as the connected component of  $\mathcal{IX}$  whose objects consist of pairs  $(x, \text{id})$ , where  $x$  is an object of  $\mathcal{X}$  and  $\text{id}$  is the identity automorphism of  $x$ , then the age of  $\mathcal{X}$  is 0 and  $A^*(|\mathcal{X}|, \mathbb{Q})$  is a direct summand of  $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$ .

Continuing with the notation of the introduction, recall that  $P$  is a  $d$ -dimensional lattice polytope in a lattice  $N$  of rank  $d$  and  $\mathcal{T}$  is a regular lattice triangulation of  $P$ . Recall that  $\mathcal{T}$  induces a fan refinement  $\Delta$  of the cone  $\sigma$  over  $P \times \{1\}$  in  $N_{\mathbb{R}} \times \mathbb{R}$ , and that  $Y = Y(\Delta)$  is the associated  $(d+1)$ -dimensional, semi-projective, simplicial toric variety. There is a canonical Deligne-Mumford stack  $\mathcal{Y}$  with coarse moduli space  $Y$  [4]. If  $F$  is a non-empty face of  $\mathcal{T}$  with vertices  $v_1, \dots, v_s$ , then set

$$\text{Box}(F) = \{w \in N_{\mathbb{R}} \times \mathbb{R} \mid w = \sum_{i=1}^s q_i(v_i, 1) \text{ for some } 0 < q_i < 1\},$$

and let  $\text{Box}(\emptyset) = \{0 \in N_{\mathbb{R}} \times \mathbb{R}\}$ . Borisov, Chen and Smith [4] showed that the inertia stack of  $\mathcal{Y}$  decomposes into connected components as  $\mathcal{IY} = \coprod_{F \in \mathcal{T}} \coprod_{w \in \text{Box}(F) \cap (N \times \mathbb{Z})} \mathcal{Y}_w$ , where  $\mathcal{Y}_w = \mathcal{Y}$  if  $w = 0$  and, if  $w \neq 0$ , then  $|\mathcal{Y}_w|$  is isomorphic to the torus-invariant subvariety  $V(F)$  of  $Y$  corresponding to the cone over  $F \times \{1\}$  in  $\Delta$ . Moreover, if  $\psi : N_{\mathbb{R}} \times \mathbb{R} \rightarrow \mathbb{R}$  denotes projection onto the second co-ordinate, then the age of  $\mathcal{Y}_w$  is  $\psi(w) \in \mathbb{Z}$ .

Recall that if  $Q$  is a lattice polytope contained in  $P$ , then  $N'$  is the intersection of  $N$  with the affine span of  $Q$  and the fan  $\Delta$  restricts to a fan  $\Sigma$  refining the cone  $\sigma'$  over  $Q \times \{1\}$  in  $(N')_{\mathbb{R}} \times \mathbb{R}$ . If  $\mathcal{Y}'$  denotes the canonical Deligne-Mumford stack with coarse moduli space  $Y' = Y'(\Sigma)$ , then the inclusion of  $N'$  in  $N$  induces an inclusion of  $\mathcal{Y}'$  as a closed substack of  $\mathcal{Y} \times (\mathbb{C}^*)^{\dim P - \dim Q}$  and an inclusion of  $\mathcal{Y}' \times (\mathbb{C}^*)^{\dim P - \dim Q}$  as an open substack of  $\mathcal{Y}$ . These inclusions induce a corresponding restriction map  $\iota : A_{\text{orb}}^*(\mathcal{Y}, \mathbb{Q}) \rightarrow A_{\text{orb}}^*(\mathcal{Y}', \mathbb{Q})$ , which we describe below (c.f. Remark 2.2).

If  $\mathcal{T}|_Q$  denotes the restriction of  $\mathcal{T}$  to  $Q$ , then the inertia stack of  $\mathcal{Y}'$  decomposes into connected components as  $\mathcal{IY}' = \coprod_{F \in \mathcal{T}|_Q} \coprod_{w \in \text{Box}(F) \cap (N \times \mathbb{Z})} \mathcal{Y}'_w$ , where  $\mathcal{Y}'_w = \mathcal{Y}'$  if  $w = 0$  and, if  $w \neq 0$ , then the age of  $\mathcal{Y}'_w$  is  $\psi(w)$  and  $|\mathcal{Y}'_w|$  is isomorphic to the torus-invariant subvariety  $V(F)'$  of  $Y'$  corresponding to the cone over  $F \times \{1\}$  in  $\Sigma$ . For each face  $F \in \mathcal{T}|_Q$ , the inclusion of  $N'$  in  $N$  induces a closed immersion  $V(F)' \hookrightarrow V(F)' \times (\mathbb{C}^*)^{\dim P - \dim Q}$  and an open immersion  $V(F)' \times (\mathbb{C}^*)^{\dim P - \dim Q} \hookrightarrow V(F)$ . The corresponding restriction map  $\nu_F : A^*(V(F), \mathbb{Q}) \rightarrow A^*(V(F)', \mathbb{Q})$  is surjective since if  $W'$  is an irreducible closed subvariety of  $V(F)'$  and  $W$  denotes the closure

of  $W' \times (\mathbb{C}^*)^{\dim P - \dim Q}$  in  $V(F)$ , then  $\nu_F([W]) = [W']$ . The restriction map  $\iota : A_{\text{orb}}^*(\mathcal{Y}, \mathbb{Q}) \rightarrow A_{\text{orb}}^*(\mathcal{Y}', \mathbb{Q})$  has the form

$$\iota : \coprod_{F \in \mathcal{T}} \coprod_{w \in \text{BOX}(F) \cap (N \times \mathbb{Z})} A^*(|\mathcal{Y}_w|, \mathbb{Q})[\psi(w)] \rightarrow \coprod_{F \in \mathcal{T}|_Q} \coprod_{w \in \text{BOX}(F) \cap (N \times \mathbb{Z})} A^*(|\mathcal{Y}'_w|, \mathbb{Q})[\psi(w)],$$

where for each  $F \in \mathcal{T}$  and  $w \in \text{BOX}(F) \cap (N \times \mathbb{Z})$ ,  $\iota$  restricts to  $\nu_F$  (with a grading shift) on  $A^*(|\mathcal{Y}_w|, \mathbb{Q})[\psi(w)]$  if  $F \subseteq Q$  and restricts to zero otherwise. One can verify from the description of the ring structure of an orbifold Chow ring in [1] that  $\iota$  is a ring homomorphism. We conclude that  $\iota$  is a surjective ring homomorphism, thus establishing Lemma 1.2.

*Remark 2.1.* The dimensions of the graded pieces of  $A^*(V(F), \mathbb{Q})$  are equal to the coefficients of an  $h$ -vector of a fan [8, Corollary 2.12]. The analogous combinatorial proof of Stanley's theorem goes as follows: one can express  $\delta_P(t)$  and  $\delta_Q(t)$  as sums of shifted  $h$ -vectors [3, 11], and then apply Stanley's monotonicity theorem for  $h$ -vectors [13] to conclude the result.

*Remark 2.2.* Consider the deformed group ring  $\mathbb{Q}[N \times \mathbb{Z}]^\Delta := \oplus_{v \in \sigma \cap (N \times \mathbb{Z})} \mathbb{Q} \cdot y^v$ , with ring structure defined by

$$y^v \cdot y^w = \begin{cases} y^{v+w} & \text{if there exists a cone } \tau \in \Delta \text{ containing } v \text{ and } w \\ 0 & \text{otherwise.} \end{cases}$$

If  $v_1, \dots, v_t$  denote the vertices of  $\mathcal{T}$  and  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ , then Jiang and Tseng [9, Theorem 1.1] showed that there is an isomorphism of rings

$$A_{\text{orb}}^*(Y, \mathbb{Q}) \cong \frac{\mathbb{Q}[N \times \mathbb{Z}]^\Delta}{\{\sum_{i=1}^t \langle (v_i, 1), u \rangle y^{(v_i, 1)} \mid u \in M \times \mathbb{Z}\}}.$$

Similarly, if  $v_1, \dots, v_s$  are the vertices of  $\mathcal{T}|_Q$  and  $M' = \text{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$ , then

$$A_{\text{orb}}^*(Y', \mathbb{Q}) \cong \frac{\mathbb{Q}[N' \times \mathbb{Z}]^\Sigma}{\{\sum_{i=1}^s \langle (v_i, 1), u \rangle y^{(v_i, 1)} \mid u \in M' \times \mathbb{Z}\}}.$$

Consider the surjective ring homomorphism  $j : \mathbb{Q}[N \times \mathbb{Z}]^\Delta \rightarrow \mathbb{Q}[N' \times \mathbb{Z}]^\Sigma$  satisfying  $j(y^v) = y^v$  if  $v \in \Sigma$  and  $j(y^v) = 0$  if  $v \notin \Sigma$ . The induced ring homomorphism

$$\frac{\mathbb{Q}[N \times \mathbb{Z}]^\Delta}{\{\sum_{i=1}^t \langle (v_i, 1), u \rangle y^{(v_i, 1)} \mid u \in M \times \mathbb{Z}\}} \longrightarrow \frac{\mathbb{Q}[N' \times \mathbb{Z}]^\Sigma}{\{\sum_{i=1}^s \langle (v_i, 1), u \rangle y^{(v_i, 1)} \mid u \in M' \times \mathbb{Z}\}}$$

corresponds to the restriction map  $\iota : A_{\text{orb}}^*(Y, \mathbb{Q}) \rightarrow A_{\text{orb}}^*(Y', \mathbb{Q})$  under the above isomorphisms. The existence of such a ring homomorphism was used by Stanley in his original commutative algebra proof of Theorem 1.1 [13].

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